Convergence Analysis of the Geometric Thin-Film Equation

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The Geometric Thin-Film Equation

Fix $\alpha > 0$. The **Geometric Thin-Film Equation** is given by

$$\partial_t h + \partial_x (h\bar{h}^2 \partial_{xxx}\bar{h}) = 0 \qquad \text{where} \qquad \bar{h} = K * h \tag{GTFE}$$

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Here, the measure-valued function $h : \mathbb{R}^+ \to \mathcal{M}(\mathbb{R})$ represents the basic free-surface height,

$$K(x) := \frac{1}{4\alpha^2}(\alpha + |x|)\mathrm{e}^{-|x|/\alpha}$$

is the Green's function for the bi-Helmholtz problem $(1 - \alpha^2\partial_{xx})^2 K(x) = \delta(x)$, and

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Ó Náraigh and Pang introduced (GTFE) as a novel regularization of the **thin-film equation**

$$\partial_t h = -\partial_x(h^3\partial_{xxx}h)$$

in one spatial variable.

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\begin{aligned}
\int_{-\infty}^{\infty} \phi(t,0)\,\mathrm{d}h(0) + \int_0^{\infty}\int_{-\infty}^{\infty} \bar{h}(1-\alpha^2\partial_{xx})^2\phi_t\,\mathrm{d}x\,\mathrm{d}t + \int_0^{\infty}\int_{-\infty}^{\infty} \bar{h}^3\partial_{xxx}\bar{h}\phi_x\,\mathrm{d}x\,\mathrm{d}t \\
-2\alpha^2\int_0^{\infty}\int_{-\infty}^{\infty} \bar{h}^2\partial_{xx}\bar{h}\partial_{xxx}\bar{h}\phi_x\,\mathrm{d}x\,\mathrm{d}t - \alpha^4\int_0^{\infty}\int_{-\infty}^{\infty} \bar{h}\partial_x\bar{h}(\partial_{xxx}\bar{h})^2\phi_x\,\mathrm{d}x\,\mathrm{d}t \\
-\frac{1}{2}\alpha^4\int_0^{\infty}\int_{-\infty}^{\infty} (\bar{h}\partial_{xxx}\bar{h})^2\phi_{xx}\,\mathrm{d}x\,\mathrm{d}t = 0, \qquad \text{(W)}
\end{aligned}
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for all indefinitely differentiable and compactly-supported test functions $\phi \in C_c^{\infty}(\Omega)$.

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1. Given initial data $h(0) = \mu \in B_{\mathcal{M}^+(\mathbb{R})}$, does (W) have a solution $(h, \bar{h})$ in Ω?
2. If so, what is the regularity of $\bar{h}$?
3. Finally, to what extent is the solution unique?

A potential difficulty!

Equation (W) involves derivatives of order three and above, but K is only twice classically differentiable and

$$K'''(x) = \frac{1}{4\alpha^2}\left(\frac{2\,\mathrm{sgn}(x)}{\alpha^2} - \frac{x}{\alpha^3}\right)\mathrm{e}^{-\frac{|x|}{\alpha}}, \qquad x \neq 0,$$

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is undefined and has no limit at the origin.

Loosely speaking, it follows that:

1. standard existence theorems requiring Lipschitz or continuous kernels don't apply;
2. ideally solutions should avoid this point of discontinuity to prevent bad behaviour.

Our approach to establishing existence

1. Write $h(t)$ as a push-forward measure

$$h(t) = c(t,\cdot)_*\mu, \qquad \int_{-\infty}^{\infty} f(x)\,\mathrm{d}h(t)(x) = \int_{-\infty}^{\infty} f(c(t,x))\,\mathrm{d}\mu(x),$$

for some Borel function $c : \mathbb{R}^+ \times M \to \mathbb{R}$ to be determined, where M is a closed set such that $\operatorname{supp}\mu \subseteq M \subseteq \mathbb{R}$.

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2. To avoid problems with K''' at the origin, require that c satisfies a 'no-crossing' condition:

$$c(t,x) < c(t,y) \qquad \text{whenever } x, y \in M, x < y, \tag{NC}$$

i.e. $c(t,\cdot)$ is strictly increasing on M for all $t \in \mathbb{R}^+$.

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4. Prove that the ODE system has a solution c satisfying (NC).

The ODE system

Theorem 1

Let $c : \mathbb{R}^+ \times M \to \mathbb{R}$ satisfy (NC) and

$$\begin{cases} c_t(t,x) &= \bar{h}(t,c(t,x))^2 \displaystyle\int_{z\neq x} K'''(c(t,x)-c(t,z))\,\mathrm{d}\mu(z) \\ c(0,x) &= x \end{cases}, \quad (t,x) \in (0,\infty)\times M. \tag{ODE}$$

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4. As $c(t,\cdot)$ is strictly increasing, $c(t,x) - c(t,z) \neq 0$ for $z \neq x$, so we avoid the undefined $K'''(0)$.

Solutions of (ODE) step 1: particle solutions

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Then (ODE) becomes

$$\begin{cases} \dot{x}_i(t) &= \left(\sum_{j=1}^N w_j K(x_i(t) - x_j(t))\right)^2 \left(\sum_{j \neq i} w_j K'''(x_i(t) - x_j(t))\right), \\ x_i(0) &= a_i \end{cases} \quad t > 0,\ 1 \leq i \leq N. \qquad \text{(FODE)}$$

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Lemma 2

Define the open set $D = \{\boldsymbol{x} \in \mathbb{R}^N : x_i < x_j, i < j\}$. Then the velocity field $\boldsymbol{v} : D \to \mathbb{R}^N$ given by

$$v_i(\boldsymbol{x}) = \left(\sum_{j=1}^N w_j K(x_i - x_j)\right)^2 \left(\sum_{j \neq i} w_j K'''(x_i - x_j)\right)$$

is Lipschitz.

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Proposition 3

There exists a unique solution $\mathbf{x} : \mathbb{R}^+ \to D$ of (FODE).

Equivalently, $c : \mathbb{R}^+ \times M \to \mathbb{R}$, given by $c(t, a_i) = x_i(t)$, $1 \leq i \leq N$, is the unique solution of (ODE) with initial data $\mu = \sum_{i=1}^{N} w_i \delta_{a_i}$.

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1. (FODE) is equivalent to $\dot{\boldsymbol{x}} = \boldsymbol{v}(\boldsymbol{x})$, $\boldsymbol{x}(0) = (a_1, \ldots, a_N) \in D$, so as $\boldsymbol{v}$ is Lipschitz, a standard application of the Picard-Lindelöf theorem yields a unique local solution for small time t.

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2. To get a unique **global** solution, we must show that D is a **trapping region**, i.e. one in which no solution that starts in D can escape D in finite time.
3. We will call any solution $c : \mathbb{R}^+ \times M \to \mathbb{R}$ furnished by Proposition 3 a **particle solution**.

Solutions of (ODE) step 2: convergence of particle solutions

Theorem 4

Given $\mu = B_{\mathcal{M}^+(\mathbb{R})}$, there exists an increasing sequence of finite sets $M_N \subseteq \mathbb{R}$ and measures μ_N, $\operatorname{supp} \mu_N \subseteq M_N$, such that the corresponding sequence of particle solutions $c_N : \mathbb{R}^+ \times M_N \to \mathbb{R}$ 'converges' to $c : \Omega \to \mathbb{R}$ that satisfies (NC) and solves (ODE) with initial data μ.

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1. The μ_N satisfy $\mu_N \overset{w^*}{\to} \mu$ (via the identification $\mathcal{M}(\mathbb{R}) = C_0(\mathbb{R})^*$), but some stronger convergence behaviour is needed in the proof.

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2. The proof is reminiscent of that of Helly's selection theorem.
3. Some careful estimates (established by analysing K, K''' and using Grönwall's Lemma) are needed to ensure that c satisfies (NC).
4. Further estimates (obtained using the 'strong' convergence mentioned above) are needed to show that the sequence of sums in (FODE) converge to the integral in (ODE).

Regularity of solutions

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2. The $\frac{1}{2}$-Hölder continuity follows from the fact that $h : \mathbb{R}^+ \to B_{\mathcal{M}(\mathbb{R})}$ is Lipschitz with respect to a Wasserstein-like metric.

Uniqueness of solutions

Theorem 6

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1. First a local uniqueness result is established, followed by the global one.
2. The local result is obtained by considering a map on a suitable Fréchet space having contraction-like properties.
3. There exist solutions $h \neq h'$ of (W) both having initial data δ_0, that (necessarily) are not both of the push-forward form described above. So solutions of (W) are not unique in general.

Thank you for listening! Summary of main results

Theorem 1

The basic free-surface height h solves (W) if $c : \mathbb{R}^+ \times M \to \mathbb{R}$ satisfies (NC) and solves (ODE):

$$c_t(t,x) = \bar{h}(t, c(t,x))^2 \int_{z \neq x} K'''(c(t,x) - c(t,z))\, \mathrm{d}\mu(z), \qquad c(0,x) = x, \qquad (t,x) \in (0,\infty) \times M.$$

Theorem 4

There is a function $c : \Omega \to \mathbb{R}$ that satisfies (NC) and solves (ODE) with initial data $\mu \in B_{\mathcal{M}^+(\mathbb{R})}$.

Theorem 5

Let $c : \Omega \to \mathbb{R}$ satisfy (NC) and solve (ODE). Then $\bar{h} \in C_b^{0,\frac{1}{2}}(\mathbb{R}^+; H^3(\mathbb{R}))$.

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